NOTE ON EXPLICIT PROOF OF POINCARÉ INEQUALITY FOR DIFFERENTIAL FORMS ON MANIFOLDS

LEONID SHARTSER

To the third anniversary of P. Milman's working seminar
for his graduate students and postdocs

ABSTRACT. We prove a Poincaré type inequality for differential forms on compact manifolds by means of a constructive 'globalization' of a local Poincaré inequality on convex sets.

1. INTRODUCTION

In a recent paper V. Goldshtein and M. Troyanov [GoTr] proved Sobolev-Poincaré type inequality for differential forms on compact Riemannian manifolds. In this article we present a constructive alternative method of proof. The latter allows, in particular, to estimate the constants in the inequalities in geometric terms. Namely, we construct for any smooth r-form ω on a Riemannian manifold M a smooth r-form ξ on M such that $d\omega = d\xi$ and inequality

$$(1.1) \qquad \|\xi\|_{L^p(M)} \leq C\|d\omega\|_{L^q(M)}$$

holds for p and q in a certain (standard) range with a positive constant C depending only on p, q, r and manifold the M (Theorems 2.2 and 3.1). The structure of the proof is first to show inequality (1.1) locally by means of adapting a proof of Lemma 3.11 from [BoMi] to our setting with differential forms, and then, globalizing it by means of a novel method that we present in Section 3.

We are mainly interested in Poincaré type inequalities due to the geometric information that they encode. Our primary goal is to study such inequalities on singular sets of algebraic nature, such as semialgebraic sets, in order to better understand the metric behavior of such sets. Constructive proofs of such inequalities would, hopefully, allow to extend results of this type to a singular setting. The results of this article were announced in [S].

Throughout this chapter we will use the following notations.

Notation 1.1.

- Suppose that X is a set and $f, g : X \to \mathbb{R}$ are two functions. We write $f \lesssim g$ if there exists a positive constant C such that $f \leq Cg$.
- The symbol $\mathbb{N}$ will denote the set of natural numbers $\{1, 2, \dots\}$.
- If A is a measurable subset of $\mathbb{R}^n$ we write $\mathrm{Vol}(A)$ to denote its n-dimensional volume.
- If $p > 1$ denote by p' its Hölder conjugate, that is, $1/p + 1/p' = 1$.
- If $x, y \in \mathbb{R}^n$, we write $d(x, y) := |x - y|$.

2. Local Poincaré inequality

In this section we prove a local Poincaré type inequality for differential forms. That is, we prove inequality (1.1) with M being a convex set. This inequality is well known and was studied, e.g. , in [IwLu]. Our proof of local inequality (1.1) utilizes a slightly different approach from the one used in [IwLu]. We show that Poincaré inequality for differential forms (Theorem 2.2) is a simple consequence of 'universal' inequality (Proposition 2.1) that extends Lemma 3.11 from [BoMi] to differential forms.

Suppose that M is an orientable Riemannian manifold. We denote by $\Omega^\bullet(M)$ the algebra of smooth differentiable forms on M. Define an L^p norm of a form $\omega \in \Omega^r(M)$ by $\|\omega\|_{L^p} := \left(\int_M |\omega|^p d\mathrm{Vol}\right)^{1/p}$ where $|\omega|$ denotes the pointwise norm of ω and $d\mathrm{Vol}$ denotes the volume form on M.

2.1. **Poincaré inequality on a convex set in $\mathbb{R}^n$.** Let $D \subset \mathbb{R}^n$ be a convex set. For each $y \in D$ define a homotopy operator

$$K_y : \Omega^r(D) \to \Omega^{r-1}(D)$$

by the following formula.

$$K_y\omega := \int_0^1 \psi_y^* \omega \, dt,$$

where $\psi_y : D \times [0, 1] \to D$, $\psi_y(x, t) := tx + (1 - t)y$. It is easy to check that $dK_y\omega + K_y d\omega = \omega$.

The next proposition is an extension of Lemma 3.11 from [BoMi] to differential forms with nearly the same proof, i.e. by interchanging the order of integrations on the left hand side of the inequality.

Proposition 2.1. *Let D be a convex compact set in $\mathbb{R}^n$, $r \in \mathbb{N} \cup \{0\}$ and $p, q \geq 1$ such that*

$$(i) \quad p \geq q \text{ and } \frac{1}{q} - \frac{1}{p} < \frac{1}{n}$$

or

$$(ii) \quad p < q.$$

Then,

$$\left\| \frac{1}{\mathrm{Vol}(D)^{1/p}} \left\| \frac{K_y d\omega(x)}{d(x,y)} \right\|_{L^q(dy)} \right\|_{L^p(dx)} \leq C(p,q,r,n) \left\| d\omega \right\|_{L^q(dx)}$$

for any r-form ω, where

$$C(p,q,r,n) := \begin{cases} \int_0^1 \min(t^{n/p}, (1-t)^{n/p}) t^{r-n/p}(1-t)^{-n/q} dt & \text{in case } (i) \\ \int_0^1 \min(t^{n/q}, (1-t)^{n/q}) t^{r-n/q}(1-t)^{-n/q} dt & \text{in case } (ii) \end{cases}$$

Proof. For a *k*-form α the pullback $\psi_y^* \alpha(x,t)$ can be written in the form

$$\alpha_0 + dt \wedge \alpha_1.$$

Denote by $|\psi_y^* \alpha(x,t)|_1$ the pointwise norm $|\alpha_1(x,t)|$.
Suppose that $p \geq q$.

$$
\begin{aligned}
\left\| \left\| \frac{K_y d\omega(x)}{|x-y|} \right\|_{L^q(dy)} \right\|_{L^p(dx)} &= \left\{ \int_D \left\| \frac{K_y d\omega(x)}{|x-y|} \right\|_{L^q(dy)}^p dx \right\}^{1/p} \\
&= \left\{ \int_D \left(\int_D \left| \int_0^1 \frac{\psi_y^* d\omega(x,t)}{|x-y|} dt \right|^q dy \right)^{p/q} dx \right\}^{1/p} \\
&\leq \left\{ \int_D \left(\int_D \left[\int_0^1 \frac{|\psi_y^* d\omega(x,t)|_1}{|x-y|} dt \right]^q dy \right)^{p/q} dx \right\}^{1/p} \\
&\leq \left\{ \int_D \left(\int_0^1 \left[\int_D \frac{|\psi_y^* d\omega(x,t)|_1^q}{|x-y|^q} dy \right]^{1/q} dt \right)^p dx \right\}^{1/p} \\
&\leq \int_0^1 \left\{ \int_D \left[\int_D \frac{|\psi_y^* d\omega(x,t)|_1^q}{|x-y|^q} dy \right]^{p/q} dx \right\}^{1/p} dt.
\end{aligned}
$$

Observe that if $d\omega_1(x,t)$ is the component of $\psi_y^* d\omega(x,t)$ that contains dt, then for a collection of vectors $\xi_1, \ldots, \xi_k$ we have

$$d\omega_1(x,t)(\xi_1, \ldots, \xi_k) = t^k d\omega(\psi_y(x,t); x-y, \xi_1, \ldots, \xi_k).$$

It follows that

$$\left| \psi_y^* d\omega(x,t) \right|_1 \leq t^k |x-y| \, |d\omega(\psi_y(x,t))|.$$

Therefore,

$$\int_0^1 \left\{ \int_D \left[\int_D \frac{|\psi_y^* d\omega(x,t)|_1^q}{|x-y|^q} dy \right]^{p/q} dx \right\}^{1/p} dt \quad \le$$

(2.2)
$$\int_0^1 \left\{ \int_D \left[\int_D t^{kq} |d\omega(\psi_y(x,t))|^q \, dy \right]^{p/q} dx \right\}^{1/p} dt \quad =$$

Set $u = tx$ and $v = (1-t)y$ obtaining $du = t^n dx$ and $dv = (1-t)^n dy$

$$\int_0^1 \left\{ \int_{tD} \left[\int_{(1-t)D} |d\omega(u+v)|^q \, dv \right]^{p/q} du \right\}^{1/p} t^{k-n/p}(1-t)^{-n/q} dt \quad =$$

set $z = u + v$,

$$\int_0^1 \left\{ \int_{tD} \left[\int_{u+(1-t)D} |d\omega(z)|^q \, dz \right]^{p/q} du \right\}^{1/p} t^{k-n/p}(1-t)^{-n/q} dt.$$

Note that D is convex we have $u + (1-t)D \subset D$. Let us examine the expression in { }.

$$\left\{ \int_{tD} \left[\int_{u+(1-t)D} |d\omega(z)|^q \, \phi(y(z-u)) dz \right]^{p/q} du \right\}^{1/p} \quad =$$

$$\left\{ \int_{tD} \left[\int_D \mathbf{1}_{u+(1-t)D}(z) \, |d\omega(z)|^q \, dz \right]^{p/q} du \right\}^{1/p} \quad =$$

$$\left\| \int_D \mathbf{1}_{u+(1-t)D}(z) \, |d\omega(z)|^q \, dz \right\|_{L^{p/q}(tD,du)}^{1/q} \quad \le$$

$$\left(\int_D \left\| \mathbf{1}_{u+(1-t)D}(z) \, |d\omega(z)|^q \right\|_{L^{p/q}(tD,du)} dz \right)^{1/q} \quad \le$$

$$\left(\int_D |d\omega(z)|^q \left\| \mathbf{1}_{u+(1-t)D}(z) \right\|_{L^{p/q}(tD,du)} dz \right)^{1/q}.$$

Now, consider the following estimate.

$$\left\| \mathbf{1}_{u+(1-t)D}(z) \right\|_{L^{p/q}(tD,du)} \le \left\| \mathbf{1}_{u+(1-t)D}(z) \right\|_{L^{p/q}(tD,du)}$$
$$\le (\text{Vol}(D) \min(t^n, (1-t)^n))^{q/p}.$$

Summarizing all the computations we finally obtain

$$\left\| \left\| \frac{K_y d\omega(x)}{|x-y|} \right\|_{L^q(dy)} \right\|_{L^p(dx)} \leq$$

$$\leq \int_0^1 \left(\int_D |d\omega(z)|^q \left(\text{Vol}(D) \min(t^n, (1-t)^n) \right)^{q/p} dz \right)^{1/q} t^{k-n/p}(1-t)^{-n/q} dt$$

$$\leq (\text{Vol}(D))^{1/p} \int_0^1 \|d\omega\|_{L^q(D)} \min(t^{n/p}, (1-t)^{n/p}) t^{k-n/p}(1-t)^{-n/q} dt$$

$$\leq (\text{Vol}(D))^{1/p} C(p,q,k,n) \|d\omega\|_{L^q(D)}.$$

Now suppose that $p < q$. Up to equation (2.2) everything is the same. Let

$$I := \left\{ \int_D \left[\int_D |d\omega(\psi_y(x,t))|^q \, dy \right]^{p/q} dx \right\}^{1/p}$$

By Hölder inequality with exponent $r = q/p$ we have:

$$I \leq \left\{ \left(\int_D 1 dx \right)^{1/r'} \left(\int_D \left[\int_D |d\omega(\psi_y(x,t))|^q \, dy \right]^{rp/q} dx \right)^{1/r} \right\}^{1/p}$$

$$\leq \text{Vol}(D)^{\frac{1}{p}-\frac{1}{q}} \left(\int_D \left[\int_D |d\omega(\psi_y(x,t))|^q \, dy \right] dx \right)^{1/q} =$$

Set $u = tx$, $v = (1-t)y$ and $z = u + v$ obtaining $du = t^n dx$, $dv = (1-t)^n dy$ and $dz = dv$

$$= \text{Vol}(D)^{\frac{1}{p}-\frac{1}{q}} t^{-n/q}(1-t)^{-n/q} \left(\int_{tD} \int_{u+(1-t)D} |d\omega(z)|^q \, dzdu \right)^{1/q}$$

$$= \text{Vol}(D)^{\frac{1}{p}-\frac{1}{q}} t^{-n/q}(1-t)^{-n/q} \left(\int_{tD} \int_D |d\omega(z)|^q \, \mathbf{1}_{u+(1-t)D}(z) dzdu \right)^{1/q}$$

$$= \text{Vol}(D)^{\frac{1}{p}-\frac{1}{q}} t^{-n/q}(1-t)^{-n/q} \left(\int_D |d\omega(z)|^q \int_{tD} \mathbf{1}_{u+(1-t)D}(z) dudz \right)^{1/q}$$

$$\leq \text{Vol}(D)^{1/p} t^{-n/q}(1-t)^{-n/q} \min(t^{n/q}, (1-t)^{n/q}) \|d\omega\|_{L^q(D)}.$$

The inequality of the lemma follows from here. $\qquad\square$

Next, we prove the local Poincaré inequality.

Theorem 2.2. *Suppose that $p, q \geq 1$ are as in Proposition 2.1. Let ω be a smooth r-form on a convex set $D \subset \mathbb{R}^n$. There exists an r-form ξ on D such that $d\omega = d\xi$ and*

$$\|\xi\|_{L^p(D)} \leq c \|d\omega\|_{L^q(D)},$$

where $c := Vol(D)^{1/p-1/q} C(p,q,k,n) R$, with $C(p,q,k,n)$ from Proposition 2.1.

Proof. Define an average homotopy operator A by the formula

$$A\omega := \frac{1}{\mathrm{Vol}(D)} \int_D K_y \omega dy.$$

Set $\xi = Ad\omega$. Note that $dA\omega + Ad\omega = \omega$ and therefore $d\xi = d\omega$. Denote by R the diameter of D. Using Hölder inequality and Proposition 2.1 we obtain the following estimate.

$$
\begin{aligned}
\|Ad\omega\|_{L^p(D)} &= \left\| \frac{1}{\mathrm{Vol}(D)} \int_D K_y d\omega(x) dy \right\|_{L^p(D,dx)} \\
&= \frac{1}{\mathrm{Vol}(D)} \left\| \int_D \frac{K_y d\omega(x)}{d(x,y)} d(x,y) dy \right\|_{L^p(D,dx)} \\
\text{(Hölder inequality)} \quad &\leq \frac{1}{\mathrm{Vol}(D)} \left\| \left\| \int_D \frac{K_y d\omega(x)}{d(x,y)} \right\|_{L^q(dy)} \|d(x,y)\|_{L^{q'}(dy)} \right\|_{L^p(D,dx)} \\
&\leq \frac{1}{\mathrm{Vol}(D)} \sup_{x \in D} \|d(x,y)\|_{L^{q'}(dy)} \left\| \left\| \int_D \frac{K_y d\omega(x)}{d(x,y)} \right\|_{L^q(dy)} \right\|_{L^p(D,dx)} \\
\text{(Proposition 2.1)} \quad &\leq \mathrm{Vol}(D)^{1/p-1/q} c(p,q,k,n) R \|d\omega\|_{L^q(D)}.
\end{aligned}
$$

$\square$

3. Globalization of Poincaré type inequality

In this section we describe a constructive method of proof of Poincaré type inequality on a compact manifold. The idea of our construction was inspired from the construction of double Čech-De Rham complex (see [BT]).

The main Theorem of this section is

Theorem 3.1. *(Global Poincaré Inequality) Let M be a compact Riemannian manifold and ω an exact r-form on it. Suppose that p and q are as in Proposition 2.1. There exists an $(r-1)$-form ξ on M such that*

$$(3.3) \qquad\qquad d\xi = \omega \quad and \quad \|\xi\|_{L^p(M)} \lesssim \|\omega\|_{L^q(M)}.$$

In what follows we describe the construction of the form ξ from the latter Theorem. We begin with some basic definitions.

Definitions 3.2. Let M be a Riemannian manifold. A subset $D \subset M$ is called **convex** if for every two points $p, q \in D$ there exists a unique geodesic that connects p with q and lies entirely in D. Let $\mathcal{U} = \{U_i\}$ be a cover of M. Denote by U_I the set $U_{i_0} \cap \cdots \cap U_{i_s}$ where $I = (i_0, \ldots, i_s)$. The cover $\mathcal{U}$ is called **good cover** if every $U \in \mathcal{U}$ is convex. The **nerve complex** of $\mathcal{U}$ is a simplicial complex $(C_j(\mathcal{U}), \partial)$ generated by $\{[I] : U_I \neq \emptyset, I = (i_0, \ldots, i_j)\}$ where $\partial : C_{j+1}(\mathcal{U}) \to C_j(\mathcal{U})$ is defined by

$$\partial[(i_0, \ldots, i_{j+1})] := \sum_k (-1)^k [(i_0, \ldots, \hat{i_k} \ldots, i_{j+1})].$$

The **Čech complex** associated with the cover $\mathcal{U}$ is denoted by $(C^j(\mathcal{U}), \delta)$ where $C^j(\mathcal{U}) := Hom(C_j(\mathcal{U}), \mathbb{R})$ and $\delta := \partial^* : C^j(\mathcal{U}) \to C^{j+1}(\mathcal{U})$ is the dual operator to ∂. The kth cohomology group of $C^\bullet(\mathcal{U})$ is denoted by $H^k(C^\bullet(\mathcal{U}))$.

Remark 3.3. It is well known that sufficiently small balls in a Riemannian manifold M are convex (see [D] Proposition 4.2). Therefore, there exists a good cover $\mathcal{U}$ for M.

From here on, we will assume that we are in the setting of Theorem 3.1. Let $\mathcal{U} := \{U_i\}$, $i = 1, \ldots, N$ be a good cover of M. In the definition below we define the Čech complex associated with the sheaf of smooth r-forms on M.

Definition 3.4. Set

$$K^{r,0} := \Omega^r(M), \quad K^{r,s} := \bigoplus_{i_0 < \cdots < i_{s-1}} \Omega^r(U_I)$$

Let $\alpha \in K^{r,s}$. Denote by α_I, $I = (i_0, \ldots, i_{s-1})$ the components of α. Define $\delta : K^{r,s} \to K^{r,s+1}$, $s \geq 0$,

$$(\delta\alpha)_J := \left(\sum_{t=0}^{s} (-1)^t \alpha_{j_0 \ldots \hat{j}_t \ldots j_s} \right) \Bigg|_{U_J}, \quad J = (j_0, \ldots, j_s) \ .$$

Define an L^p norm on $K^{r,s}$ as follows.

$$\|\alpha\|_{L^p(K^{r,s})} := \sum_{i_0 < \cdots < i_{s-1}} \|\alpha_I\|_{L^p(U_I)}.$$

Convention 3.5. We will use the following convention. If $\alpha \in K^{r,s}$ with components α_I, $I = (i_0, \ldots, i_{s-1})$, $i_0 < \cdots < i_{s-1}$ and τ is a permutation of $\{0, \ldots s-1\}$ then $\alpha_I = \alpha_{\tau(I)} sign(\tau)$.

In the next proposition we list fundamental properties of the complex $(K^{r,\bullet}, \delta)$.

Proposition 3.6.
  i. $(K^{r,\bullet}, \delta)$ is a complex, i.e. $\delta^2 = 0$.
  ii. $(K^{r,\bullet}, \delta)$ is an exact complex, i.e. δ cohomology of $(K^{r,\bullet}, \delta)$ are trivial and moreover, if $\beta \in K^{r,s+1}$ with $\delta\beta = 0$ then there exists $\alpha \in K^{r,s}$ such that $\beta = \delta\alpha$ and
    - $\|\alpha\|_{L^p(K^{r,s})} \lesssim \|\beta\|_{L^p(K^{r,s+1})}$,
    - $\|d\alpha\|_{L^p(K^{r+1,s})} \lesssim \|\beta\|_{L^p(K^{r,s+1})} + \|d\beta\|_{L^p(K^{r+1,s+1})}$.

Proof. The proof of this proposition without estimates can be found in [BT] Proposition 8.3 and 8.5. Part (i) follows from a direct computation of δ^2. For part (ii), suppose that $\beta \in K^{r,s+1}$, $\delta\beta = 0$. Let ρ_j be a partition of unity subordinate to the cover $\{U_i\}$. Set

$$(3.4) \qquad \alpha_{i_0,\ldots,i_{s-1}} := \sum_j \rho_j \beta_{j,i_0,\ldots,i_{s-1}}.$$

Direct computation shows that $\delta\alpha = \beta$.

Now the first estimate can be obtained as follows.

$$
\begin{aligned}
\|\alpha\|_{L^p(K^{r,s})} &= \sum_{i_0<\cdots<i_{s-1}} \|\sum_j \rho_j \beta_{j,i_0,\ldots,i_{s-1}}\|_{L^p(U_I)} \\
&\leq \sum_{i_0<\cdots<i_{s-1}} \sum_j \|\rho_j \beta_{j,i_0,\ldots,i_{s-1}}\|_{L^p(U_I)} \\
&\leq \sum_{i_0<\cdots<i_{s-1}} \sum_j \|\beta_{j,i_0,\ldots,i_{s-1}}\|_{L^p(U_{j,I})} \\
&= \|\beta\|_{L^p(K^{r,s+1})}.
\end{aligned}
$$

And for the second estimate,

$$
\begin{aligned}
\|d\alpha\|_{L^p(K^{r+1,s})} &= \sum_{i_0<\cdots<i_{s-1}} \|\sum_j d\rho_j \wedge \beta_{j,i_0,\ldots,i_{s-1}} + \rho_j \wedge d\beta_{j,i_0,\ldots,i_{s-1}}\|_{L^p(U_I)} \\
&\leq \sum_{i_0<\cdots<i_{s-1}} \sum_j \|d\rho_j \wedge \beta_{j,i_0,\ldots,i_{s-1}}\|_{L^p(U_I)} + \|\rho_j d\beta_{j,i_0,\ldots,i_{s-1}}\|_{L^p(U_I)} \\
&\lesssim \sum_{i_0<\cdots<i_{s-1}} \sum_j \|\beta_{j,i_0,\ldots,i_{s-1}}\|_{L^p(U_{j,I})} + \|d\beta_{j,i_0,\ldots,i_{s-1}}\|_{L^p(U_{j,I})} \\
&= \|\beta\|_{L^p(K^{r,s+1})} + \|d\beta\|_{L^p(K^{r+1,s+1})}.
\end{aligned}
$$

$\square$

Before we give the general construction of the form ξ that satisfies 3.3 we illustrate the construction on an example.

Example 3.7. Suppose that ω is a closed 2 form on M. Consider the following table.

TABLE 1. Construction of ξ^k

2	$\omega \rightarrow$	ω_{i_0}		
1		$\uparrow$ $\xi^0_{i_0} \rightarrow$	$(\delta\xi^0)_{i_0,i_1}$	
$d \uparrow$ 0			$\uparrow$ $\xi^1_{i_0,i_1} \rightarrow$	$(\delta\xi^1)_{i_0,i_1,i_2}$
	$\Omega^\bullet(M)$	$\bigoplus_{i_0} \Omega^\bullet(U_{i_0})$	$\bigoplus_{i_0,i_1} \Omega^\bullet(U_{i_0,i_1})$	$\bigoplus_{i_0,i_1,i_2} \Omega^\bullet(U_{i_0,i_1,i_2})$

$\delta \quad \rightarrow$

An entry in the table represents the components of an element in the space indicated in the the same column at bottom row. The vertical arrows represent the exterior derivative d and the horizontal arrows represent the action of differential δ. Start off by placing ω in the first column of the table in the second row (corresponding to the degree of the form). Apply δ to ω to obtain an element $\oplus\omega_{i_0}$

in the second column of the table. Since $d\omega_{i_0} = 0$ and $\mathcal{U}$ is a good cover, we can apply the local Poincaré inequality to obtain an element $\xi^0 := \oplus\xi_{i_0}^0$ such that

$$d\xi_{i_0}^0 = \omega_{i_0}$$

and

$$\|\xi_{i_0}^0\|_{L^p(U_{i_0})} \lesssim \|\omega\|_{L^q(U_{i_0})} \leq \|\omega\|_{L^q(M)} \quad \text{for all } i_0.$$

Next, we apply δ to ξ^0 to get an element $\delta\xi^0 := \oplus(\delta\xi^0)_{i_0,i_1}$. Observe that

$$d\delta\xi^0 = \delta d\xi^0 = \delta\delta^0\omega = 0.$$

Therefore, once again, we can apply local Poincaré inequality to $\delta\xi^0$ to obtain an element $\xi^1 := \oplus\xi_{i_0,i_1}^1$ such that

$$d\xi^1 = \delta\xi^0$$

and

$$\|\xi_{i_0,i_1}^1\|_{L^p(U_{i_0,i_1})} \lesssim \|(\delta\xi^0)_{i_0,i_1}\|_{L^p(U_{i_0,i_1})} \quad \leq \quad \|\xi_{i_0}^0\|_{L^p(U_{i_0})} + \|\xi_{i_1}^0\|_{L^p(U_{i_1})}$$
$$\leq \quad 2\|\omega\|_{L^q(M)},$$

for all i_0, i_1. Finally, note that

$$d\delta\xi^1 = \delta d\xi^1 = \delta\delta\xi^0 = 0.$$

Since the components $(\delta\xi^1)_{i_0,i_1,i_2}$ of $\delta\xi^1$ are functions with zero exterior derivatives it follows that they are constants. So far we have only used the fact that ω is closed. In order to find a global form ξ that satisfies (3.3) we have to assume that ω is exact. Therefore, in the next step of the construction we assume that ω is exact and find a global $(r-1)$ form ξ that satisfies (3.3). By Theorem 3.10 below, there exists an element $c \in \bigoplus \Omega^0(U_{i_0,i_1})$ with constant components c_{i_0,i_1} for all i_0, i_1 such that

$$\delta\xi^1 - \delta c = \delta(\xi^1 - c) = 0.$$

Moreover, by Corollary 3.11 we have

$$\|c_{i_0,i_1}\|_{L^p(U_{i_0,i_1})} \lesssim \|\omega\|_{L^q(M)}.$$

We will construct (inductively) elements $x^1 \in \bigoplus \Omega^0(U_{i_0})$ and $x^0 \in \Omega^1(M)$ such that $\xi := x^0$ satisfies (3.3), see Table 2 below. Note that each row r of the latter table is the complex $(K^{r,\bullet}, \delta)$. By Proposition 3.6 each such row is exact. Therefore, by the same proposition, there exists an element x^1 such that $\delta x^1 = \xi^1 - c$ and the following estimates hold

$$\|x^1\|_{L^p(K^{0,1})} \lesssim \|\xi^1 - c\|_{L^p(K^{0,2})}$$

and

$$\|dx^1\|_{L^p(K^{1,1})} \lesssim \|\xi^1 - c\|_{L^p(K^{0,2})} + \|d(\xi^1 - c)\|_{L^p(K^{1,2})}.$$

TABLE 2. Construction of x^k

$d \uparrow$					
	2	$\omega \to$	ω_{i_0}		
	1	$\uparrow$	$\uparrow$		
		$x^0 \to$	$\xi^0_{i_0} - dx^1_{i_0} \to$	$0, \qquad (\delta\xi^0)_{i_0,i_1}$	
	0			$\uparrow$	
			$x^1_{i_0} \to$	$\xi^1_{i_0,i_1} - c_{i_0,i_1} \to$	0
		$\Omega^\bullet(M)$	$\bigoplus_{i_0} \Omega^\bullet(U_{i_0})$	$\bigoplus_{i_0,i_1} \Omega^\bullet(U_{i_0,i_1})$	$\bigoplus_{i_0,i_1,i_2} \Omega^\bullet(U_{i_0,i_1,i_2})$

$\delta \quad \to$

Note that

$$\delta(\xi^0 - dx^1) = d\xi^1 - d\delta x^1 = d(\xi^1 - \xi^1 + c) = 0.$$

Hence, by exactness of the second row there exists an element x^0 such that $\delta x^0 = \xi^0 - dx^1$ and we have

$$\|x^0\|_{L^p(M)} \lesssim \|\xi^0 - dx^1\|_{L^p(K^{1,1})}$$

and

$$\|dx^0\|_{L^p(M)} \lesssim \|\xi^0 - dx^1\|_{L^p(K^{1,1})} + \|d(\xi^0 - dx^1)\|_{L^p(K^{2,1})}.$$

Set $\xi := x^0$. We claim that $d\xi = \omega$. Indeed,

$$\delta(\omega - dx^0) = \delta\omega - d\delta x^0 = \delta\omega - d(\xi^0 - dx^1) = \delta\omega - d\xi^0 = 0.$$

It follows that $(\omega - dx^0)|_{U_{i_0}} = 0$ for all i_0 and therefore $\omega = dx^0$ on M. Moreover, combining all the estimates from above we obtain

$$
\begin{aligned}
\|\xi\|_{L^p(M)} &\lesssim \|\xi^0 - dx^1\|_{L^p(K^{1,1})} \\
&\lesssim \|\xi^0\|_{L^p(K^{1,1})} + \|dx^1\|_{L^p(K^{1,1})} \\
&\lesssim \|\omega\|_{L^q(M)} + \|\xi^1 - c\|_{L^p(K^{0,2})} + \|d\xi^1\|_{L^p(K^{1,2})} \\
&\lesssim \|\omega\|_{L^q(M)} + \|\xi^1\|_{L^p(K^{0,2})} + \|c\|_{L^p(K^{0,2})} + \|\delta\xi^0\|_{L^p(K^{1,2})} \\
&\lesssim \|\omega\|_{L^q(M)}.
\end{aligned}
$$

This concludes the example.

In what follows we give the general construction of the forms ξ^s and x^s as in the example above.

3.1. Construction of elements $\xi^s \in K^{r-s-1,s+1}$.

Definition 3.8. Set $\xi^{-1} := \omega$ and define ξ^s by setting the I'th component, ξ^s_I, to be a solution of the equation

$$(3.5) \qquad\qquad d\xi^s_I = (\delta\xi^{s-1})_I$$

in U_I, $I = (i_0, \ldots, i_s)$ such that

$$(3.6) \qquad\qquad \|\xi^s_I\|_{L^p(U_I)} \lesssim \|(\delta\xi^{s-1})_I\|_{L^p(U_I)},$$

for $0 \leq s \leq r - 1$.

We remark that equation (3.5) can be solved with an estimate (3.6) by means of local Poincaré inequality since U_I is convex and $d\delta\xi^{s-1} = 0$ (cf. Example 3.7). We have the following estimate of ξ^s in terms of the norm of ω:

Proposition 3.9. *Let* $I = (i_0, \ldots, i_s)$. *Then,*

$$\|\xi_I^s\|_{L^p(U_I)} \lesssim \|\omega\|_{L^q(M)}. \tag{3.7}$$

Proof. The proof is by induction on s. For $s = 0$ the statement follows from the local Poincaré inequality. Suppose that $s > 0$, we have then

$$
\begin{aligned}
\|\xi_I^s\|_{L^p(U_I)} &\lesssim \|(\delta\xi^{s-1})_I\|_{L^p(U_I)} \\
&\leq \sum_{t=0}^{s} \|\xi_{i_0\ldots\hat{i}_t\ldots i_{s-1}}^{s-1}\|_{L^p(U_I)} \\
&\leq \sum_{t=0}^{s} \|\xi_{i_0\ldots\hat{i}_t\ldots i_{s-1}}^{s-1}\|_{L^p(U_{i_0\ldots\hat{i}_t\ldots i_s})} \\
&\leq \sum_{t=0}^{s} \|\omega\|_{L^q(M)} \\
&\lesssim \|\omega\|_{L^q(M)}.
\end{aligned}
$$

$\square$

Note that ξ^{r-1} is a collection of 0-forms that satisfy $d\delta\xi^{r-1} = 0$. It means that $(\delta\xi^{r-1})_I$ are constants on each U_I, $I = (i_0, \ldots, i_r)$. (We use the same notation to denote the extension of $(\delta\xi^{r-1})_I$ to a globally defined constant function on M.)

Theorem 3.10. *There exists an element* $c \in K^{0,r}$ *with constant components* c_I, $I = (i_0, \ldots, i_{r-1})$ *such that*

$$(\delta c)_I = \sum_{t=0}^{r} (-1)^t c_{i_0,\ldots\hat{i}_t\ldots,i_r} = (\delta\xi^{r-1})_I, \quad \text{for all} \quad I = (i_0, \ldots, i_r). \tag{3.8}$$

Moreover, there exist $b_{I,L} \in \mathbb{R}$, $I = (i_0, \ldots, i_{r-1})$, $L = (l_1, \ldots, l_r)$ *such that*

$$c_I = \sum_L b_{I,L}(\delta\xi^{r-1})_L$$

where $b_{I,L}$ *depend only on the cover* $\mathcal{U}$.

We prove this theorem in subsection 3.3. As a consequence of Theorem 3.10, we obtain the following corollary.

Corollary 3.11. *The constants* c_I *from Theorem 3.10 admit the following estimate*

$$\|c_I\|_{L^p(U_I)} \leq \|\omega\|_{L^q(M)}$$

Proof. By Theorem 3.10 we may represent each c_I as $c_I = \sum_L b_{I,L}(\delta\xi^{r-1})_L$. By triangle inequality we have

$$(3.9) \qquad \|c_I\|_{L^p(U_I)} \leq \sum_L |b_{I,L}| \|(\delta\xi^{r-1})_L\|_{L^p(U_I)}$$

Observe that $(\delta\xi^{r-1})_L$ is a globally defined constant function and therefore, similarly to the proof of Proposition 3.9 we have

$$(3.10) \qquad \|(\delta\xi^{r-1})_L\|_{L^p(U_I)} = \|(\delta\xi^{r-1})_L\|_{L^p(U_L)} \left(\frac{Vol(U_I)}{Vol(U_L)}\right)^{1/p} \lesssim \|\omega\|_{L^q(M)}.$$

Now, from (3.9) and (3.10) we obtain the desired estimate.   □

3.2. **Construction of elements** $x^s \in K^{r-s-1,s}$.
The final step of the construction is to glue all the forms ξ^s, $s = 0, \ldots, r-1$ to a global solution ξ that satisfies (3.3). We construct inductively forms $x^s \in K^{r-s-1,s}$ such that $\xi := x^0$ is the desired global form (cf. Example 3.7). Set $\tilde{\xi}_I^{r-1} = \xi_I^{r-1} - c_I$ where c_I is given by Theorem 3.10 and $I = (i_0, \ldots, i_{r-1})$. Note that $d\tilde{\xi}_I^{r-1} = d\xi_I^{r-1}$ and $\delta\tilde{\xi}^{r-1} = 0$. It follows from Proposition 3.6 (ii) that there exists a form $x^{r-1} \in K^{0,r-1}$ such that $\delta x^{r-1} = \tilde{\xi}^{r-1}$ and

$$\|x^{r-1}\|_{L^p(K^{0,r-1})} \lesssim \|\tilde{\xi}^{r-1}\|_{L^p(K^{0,r})},$$

$$\|dx^{r-1}\|_{L^p(K^{1,r-1})} \lesssim \|\tilde{\xi}^{r-1}\|_{L^p(K^{0,r})} + \|d\tilde{\xi}^{r-1}\|_{L^p(K^{1,r})}.$$

It follows from Corollary 3.11 and Proposition 3.9 that

$$(3.11) \qquad \|x^{r-1}\|_{L^p(K^{0,r-1})} \lesssim \|\omega\|_{L^q(M)}$$

and

$$(3.12) \qquad \begin{aligned} \|dx^{r-1}\|_{L^p(K^{1,r-1})} &\lesssim \|\omega\|_{L^q(M)} + \|\delta\xi^{r-2}\|_{L^p(K^{1,r})} \\ &\leq \|\omega\|_{L^q(M)} + \|\xi^{r-2}\|_{L^p(K^{1,r-1})} \\ &\lesssim \|\omega\|_{L^q(M)}. \end{aligned}$$

Suppose that $x^{r-(t-1)}$ was constructed. By Proposition 3.6 (ii) there exists x^{r-t} such that

$$\delta x^{r-t} = \xi^{r-t} - dx^{r-t+1},$$

where

$$(3.13) \qquad \|x^{r-t}\|_{L^p(K^{t-1,r-t})} \lesssim \|\xi^{r-t} - dx^{r-t+1}\|_{L^p(K^{t-1,r-t+1})},$$

and

$$(3.14)$$
$$\begin{aligned} \|dx^{r-t}\|_{L^p(K^{t,r-t})} &\lesssim \|\xi^{r-t} - dx^{r-t+1}\|_{L^p(K^{t-1,r-t+1})} + \|d\xi^{r-t}\|_{L^p(K^{t,r-t+1})} \\ &\leq \|\xi^{r-t}\|_{L^p(K^{t-1,r-t+1})} + \|dx^{r-t+1}\|_{L^p(K^{t-1,r-t+1})} + \\ &\quad \|\delta\xi^{r-t-1}\|_{L^p(K^{t,r-t+1})} \\ &\lesssim \|\omega\|_{L^q(M)} + \|dx^{r-t+1}\|_{L^p(K^{t-1,r-t+1})}, \end{aligned}$$

provided that $\delta(\xi^{r-t} - dx^{r-t+1}) = 0$. Let us verify this condition:

$$
\begin{aligned}
\delta(\xi^{r-t} - dx^{r-t+1}) &= \delta\xi^{r-t} - d\delta x^{r-t+1} \\
&= \delta\xi^{r-t} - d(\xi^{r-t+1} - dx^{r-t+2}) \\
&= \delta\xi^{r-t} - d\xi^{r-t+1} \\
&= 0.
\end{aligned}
$$

Using estimates (3.11),(3.12) (3.13) and (3.14) we obtain the following proposition.

Proposition 3.12. *The forms x^s admit the following estimates:*

(1) $\|x^{r-t}\|_{L^p(K^{t-1,r-t})} \lesssim \|\omega\|_{L^q(M)}$.
(2) $\|dx^{r-t}\|_{L^p(K^{t,r-t})} \lesssim \|\omega\|_{L^q(M)}$.

Proof. The proof is by induction on t. For $t = 1$ estimates (1) and (2) are just (3.11) and (3.12). Suppose that $t > 1$. First we prove (2). By (3.14) and the induction hypothesis we have

(3.15)
$$
\begin{aligned}
\|dx^{r-t}\|_{L^P(K^{t,r-t})} &\lesssim \|\omega\|_{L^q(M)} + \|dx^{r-t+1}\|_{L^P(K^{t-1,r-t+1})} \\
&\lesssim \|\omega\|_{L^q(M)}.
\end{aligned}
$$

To prove estimate (1) we observe that from Proposition 3.9, (3.13) and (2) it follows that

$$
\begin{aligned}
\|x^{r-t}\|_{L^p(K^{t-1,r-t})} &\lesssim \|\xi^{r-t} - dx^{r-t+1}\|_{L^P(K^{t-1,r-t+1})} \\
&\lesssim \|\xi^{r-t}\|_{L^P(K^{t-1,r-t+1})} + \|dx^{r-t+1}\|_{L^P(K^{t-1,r-t+1})} \\
&\lesssim \|\omega\|_{L^q(M)}.
\end{aligned}
$$

$\square$

Finally, set $\xi := x^0$. To see that $dx^0 = \omega$ observe that

$$
\delta(\omega - dx^0) = \delta\omega - d\delta x^0 = \delta\omega - d(\xi^0 - dx^1) = \delta\omega - d\xi^0 = 0.
$$

But $\delta(\omega - dx^0)_i = (\omega - dx^0)|_{U_i}$ and therefore $\omega = dx^0$ on M. The estimate of ξ follows from Proposition 3.12 for $t = r$.

3.3. **Proof of Theorem 3.10.**

Note that the linear system of equations (3.8) has a solution if and only if $\sum_I(\delta\xi^{r-1})_I a_I = 0$ for every $a = \sum a_I[I] \in \ker\partial = \ker\delta^*$. Therefore, Theorem 3.10 is equivalent to the following proposition.

Proposition 3.13. *Let $a = \sum_I a_I[I]$ be an r-cycle then*

$$
\sum_I a_I(\delta\xi^{r-1})_I = 0.
$$

In the proof of this proposition we construct an explicit isomorphism from De Rham cohomology to Čech cohomology of a good cover from which Proposition 3.13 follows immediately. Another proof of the latter proposition is given in subsection 3.4. In this subsection we prove

Proposition 3.14. *The map*

$$Int : H^r(\Omega^\bullet(M)) \to H^r(C^\bullet(\mathcal{U}))$$

defined by

(3.16) $$\qquad\qquad (Int\ \omega)[I] := (-1)^{\lfloor \frac{r}{2} \rfloor} \delta \xi_I^{r-1},$$

where ξ_I^{r-1} is defined for the form ω as in Definition 3.8 is a well defined isomorphism.

Before we prove this proposition, we show how Proposition 3.13 follows from it.

Proof of Proposition 3.13. If ω is an exact r-form then $Int\ \omega = \delta W$. If $a = \sum a_I [I]$ is a cycle, then

$$(Int\ \omega)a = (-1)^{\lfloor \frac{r}{2} \rfloor} \sum a_I \delta \xi_I^{r-1} = (\delta W)a = W \partial a = 0.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Next, we prove Proposition 3.14.

Proof. Consider an auxiliary complex $(K^\bullet, D)$ defined by

$$K^m := \bigoplus_{r+s=m} K^{r,s+1}$$

and

$$D := d + (-1)^s \delta \quad \text{on } K^{r,s+1}.$$

This complex is called Čech-De Rham complex, see [BT] for details. Denote by $H^r(K^\bullet)$ the rth cohomology group of $K^\bullet$. The map $\delta : \Omega^r(M) \to K^r$ induces an isomorphism

$$h^0 : H^r(\Omega^\bullet(M)) \to H^r(K^\bullet)$$

([BT], Proposition 8.8). Similarly, the map $g : C^r(\mathcal{U}) \to K^r$, defined by sending an element in $C^r(\mathcal{U})$ to the corresponding element in $K^{0,r+1}$ induces an isomorphism

$$h^1 : H^r(C^\bullet(\mathcal{U})) \to H^r(K^\bullet).$$

We claim that $Int = (h^1)^{-1} \circ h^0$. Indeed, let us compute the action of $(h^1)^{-1} \circ h^0$ on closed form ω. First applying h^0 to ω we get an element defined by the D cohomology class of $\delta \omega$. Note that

$$D\xi^{s+1} = d\xi^{s+1} + (-1)^s \delta \xi^{s+1},$$

A direct computation, using the latter formula and fact that $\delta \xi^s = d\xi^{s+1}$, shows that

$$\delta \omega - \sum_{j=0}^{k} (-1)^{\lfloor \frac{j}{2} \rfloor} D\xi^j = (-1)^{\lfloor \frac{k+1}{2} \rfloor} \delta \xi^k, \quad k \geq 0.$$

It follows from here that $(h^1)^{-1} \delta^0 \omega$ is defined by the element that sends $[I]$ to $(-1)^{\lfloor \frac{r}{2} \rfloor} \delta \xi_I^{r-1}$ which is what was required to prove. $\qquad\qquad \square$

3.4. **Integral of a closed form over a chain.** The purpose of this subsection is to derive a formula that relates an integral of a closed form ω over a simplex σ with integrals of forms ξ^s (defined according to Definition 3.8) over simplices in the barycentric subdivision of σ.

Let T be a triangulation of M. Denote by $\{1, \ldots, N\}$ the set of vertices of T and set $\mathcal{U} := \{U_i\}$ where $U_i := st(i)$, the open star of vertex i. First we will show that Poincaré inequality holds near every finite intersection U_I. Then, we will show that the integral of the closed r-form ω over a cycle $a = \sum a_I[I]$ equals to $(-1)^{\lfloor \frac{r}{2} \rfloor} \sum a_I(\delta\xi^{r-1})_I$, where $[I]$ is identified here with the simplex of T with vertices $(i_0, \ldots, i_r)$. As a corollary, we immediately obtain another proof of Proposition 3.13. Indeed, the form ω is exact if and only if $\int_a \omega = 0$ for every r-cycle a of T. But if $a = \sum a_I[I]$ then

$$0 = \int_a \omega = (-1)^{\lfloor \frac{r}{2} \rfloor} \sum a_I(\delta\xi^{r-1})_I.$$

Proposition 3.15. *For every $\varepsilon > 0$ there exists a neighborhood U_I^ε of U_I such that $d(a, U_I) < \varepsilon$ for every $a \in U_I^\varepsilon$ and if ω is a closed r-form defined on U_I^ε then there exists an $(r-1)$ form ξ on U_I^ε such that $\omega = d\xi$ and*

$$\| \xi\|_{L^p(U_I^\varepsilon)} \lesssim \|\omega\|_{L^q(U_I^\varepsilon)}.$$

Proof. The set $\overline{U_I}$ can be covered by finitely many balls B_j of radius $\varepsilon/2$. Let U_I^ε be the union of those balls. Since the cover B_j is good cover one can construct elements ξ^s according to Definition 3.8. Denote by f the cochain that sends $[I]$ to $\delta\xi_I^{r-1}$ and note that it is a closed. But since U_I^ε is contractible f is exact. Therefore, there exists $(r-1)$ cochain c such that $f = \delta c$ which means that $\delta\xi^{r-1} = \delta c$. Hence, we can construct elements x^s as in subsection 3.2 such that $\xi := x^0$ is the desired form. $\qquad\square$

The derivation of a formula for an integral of a closed form ω involves barycentric subdivision of simplices. In the following definition we introduce our notations for that purpose.

Definitions 3.16. Let $\sigma = (i_0, \ldots, i_r)$ be an r-simplex, $J = (j_0, \ldots, j_r)$ be a permutation of $\{0, \ldots, r\}$ and $t \in \{1, \ldots, r\}$. The **barycenter** of an s-face $(i_{j_0}, \ldots, i_{j_s})$ of σ is denoted by $(i_{j_0}, \ldots, i_{j_s})^b$. Set

$$Sd_t(J) := \left((i_{j_0}, \ldots, i_{j_{t-1}})^b, \ldots, (i_{j_0}, \ldots, i_{j_r})^b \right).$$

Set $J' := (j_0, \ldots, j_{r-1})$. If $K = (k_0, \ldots, k_{r-1})$ is a permutation of $\{0, \ldots, r-1\}$ then define $J'_K := (j_{k_0}, \ldots, j_{k_{r-1}})$.

From now to the end of this subsection, assume that ω is a closed r-form on M and ξ^s were constructed according to Definition 3.8. In the next lemma we derive a formula for $\int_\sigma \omega$ where ω is an closed form.

Lemma 3.17. *Let ω be an exact r-form, ξ^s be defined as above and $\sigma = (i_0, \ldots, i_r)$, then*

$$\int_\sigma \omega = (-1)^{\lfloor \frac{m+1}{2} \rfloor} \sum_{J:j_0 < \cdots < j_m} sign(J) \int_{\partial Sd_{m+1}(J)} \xi^m_{i_{j_0}, \ldots, i_{j_m}}$$

$$+ (-1)^r \sum_{J:j_0 < \cdots < j_{r-1}} sign(J) \sum_{t=0}^{m-1} (-1)^{\lfloor \frac{t}{2} \rfloor} \sum_{K:k_0 < \cdots < k_t} sign(k) \int_{Sd_{t+1}(J'_K)} \xi^t_{i_{j_{k_0}}, \ldots, i_{j_{k_t}}},$$

where J runs over permutations of $\{0, \ldots, r\}$ and K runs over permutations of $\{0, \ldots, r-1\}$.

Before we prove this lemma we show how Proposition 3.13 follows from it. We will need the following combinatorial lemma.

Lemma 3.18. *Let $\sigma = (i_0, \ldots, i_r)$ be a simplex and $\xi \in K^{t,r}$ then*

$$\partial \sigma = \sum_{J:j_0 < \cdots < j_{r-1}} sign(J)(i_{j_0}, \ldots, i_{j_{r-1}}),$$

$$(\delta \xi)_{i_0, \ldots, i_r} = \sum_{J:j_0 < \cdots < j_{r-1}} sign(J) \xi_{i_{j_0}, \ldots, i_{j_{r-1}}},$$

where J runs over permutations of $\{0, \ldots, r\}$.

Proof. There exists 1:1 and onto correspondence

$$f : \{s : 0 \le s \le r\} \to \{J : J \text{ a permutation of } \{0, \ldots, r\}, \ j_0 < \cdots < j_{r-1}\},$$

defined by setting $f(s) = J$ with $j_r = s$. Since $j_0 < \cdots < j_{r-1}$, a choice of j_r determines the rest of the components of J. It is clear that f is 1:1 and onto. By applying this correspondence to the formula for $\partial \sigma$ we obtain

$$\partial \sigma = \sum_{s=0}^r (i_0, \ldots, \widehat{i_s}, \ldots i_r) = \sum_{J:j_0 < \cdots < j_{r-1}} (-1)^{j_r} (i_0, \ldots, \widehat{i_{j_r}}, \ldots, i_r)$$

$$= \sum_{J:j_0 < \cdots < j_{r-1}} (-1)^{j_r} (i_{j_0}, \ldots, \widehat{i_{j_{r-1}}}, \ldots, i_{j_r})$$

where the last equality holds since $(i_0, \ldots, \widehat{i_{j_r}}, \ldots, i_r) = (i_{j_0}, \ldots, \widehat{i_{j_{r-1}}}, \ldots, i_{j_r})$. Indeed, there exists l, $0 \le l \le r-1$ such that $j_{l-1} < j_r < j_l$. In particular, it follows that $j_s = s$ for $0 \le s \le l-1$, $j_s = s+1$ for $l \le s \le r-1$ and $j_r = l$. Observe that $sign(J)$ equals to (-1) raised the power of the number of transpositions that are needed to move j_r to its position in J which is equal to $l = j_r$. Therefore, $(-1)^{j_r} = sign(J)$. The second identity is proven the same way. $\square$

Next, we give a proof of Proposition 3.13.

Convention 3.19. In this subsection we use letters $J, J^1, \ldots$ to denote permutations of $\{0, \ldots, r\}$ letters $K, K^1, \ldots$ to denote permutations of $\{0, \ldots, r-1\}$. Components of a permutation J^l we be denoted by $(j_0^l, \ldots, j_r^l)$ (similarly for K^l). If σ is a simplex then we write $\sigma = (i_0(\sigma), \ldots, i_r(\sigma))$ or just $(i_0, \ldots, i_r)$ when there is no confusion possible.

Proof of Proposition 3.13. Using the formula in Lemma 3.17 with $m = r - 1$ and noting that $\partial Sd_r(J) = \sigma^b - (i_{j_0}, \ldots, i_{j_{r-1}})^b$ where σ^b is the barycenter of σ we obtain

$$
\begin{aligned}
\int_\sigma \omega &= (-1)^{\lfloor \frac{r}{2} \rfloor} \sum_{J: j_0 < \cdots < j_{r-1}} sign(J) \left[\xi^{r-1}_{i_{j_0}, \ldots, i_{j_{r-1}}}(\sigma^b) - \xi^{r-1}_{i_{j_0}, \ldots, i_{j_{r-1}}}((i_{j_0}, \ldots, i_{j_{r-1}})^b) \right] \\
&+ (-1)^r \sum_{J: j_0 < \cdots < j_{r-1}} sign(J) \sum_{t=0}^{r-2} (-1)^{\lfloor \frac{t}{2} \rfloor} \sum_{K: k_0 < \cdots < k_t} sign(k) \int_{Sd_{t+1}(J'_K)} \xi^t_{i_{j_{k_0}}, \ldots, i_{j_{k_t}}} \\
&= (-1)^{\lfloor \frac{r}{2} \rfloor} \left[(\delta \xi^{r-1})_{i_0, \ldots, i_r} - \sum_{J: j_0 < \cdots < j_{r-1}} sign(J) \xi^{r-1}_{i_{j_0}, \ldots, i_{j_{r-1}}} ((i_{j_0}, \ldots, i_{j_{r-1}})^b) \right] \\
&+ (-1)^r \sum_{J: j_0 < \cdots < j_{r-1}} sign(J) \sum_{t=0}^{r-2} (-1)^{\lfloor \frac{t}{2} \rfloor} \sum_{K: k_0 < \cdots < k_t} sign(k) \int_{Sd_{t+1}(J'_K)} \xi^t_{i_{j_{k_0}}, \ldots, i_{j_{k_t}}} \\
&= (-1)^{\lfloor \frac{r}{2} \rfloor} (\delta \xi^{r-1})_{i_0, \ldots, i_r} \\
&+ (-1)^r \sum_{J: j_0 < \cdots < j_{r-1}} sign(J) \sum_{t=0}^{r-1} (-1)^{\lfloor \frac{t}{2} \rfloor} \sum_{K: k_0 < \cdots < k_t} sign(k) \int_{Sd_{t+1}(J'_K)} \xi^t_{i_{j_{k_0}}, \ldots, i_{j_{k_t}}}
\end{aligned}
$$

where the second equality is obtained by application of Lemma 3.18. Set

$$
\psi(J, \sigma, \omega) := (-1)^r \sum_{t=0}^{r-1} (-1)^{\lfloor \frac{t}{2} \rfloor} \sum_{K: k_0 < \cdots < k_t} sign(k) \int_{Sd_{t+1}(J'_K)} \xi^t_{i_{j_{k_0}}, \ldots, i_{j_{k_t}}}.
$$

So we get the following identity

$$
(-1)^{\lfloor \frac{r}{2} \rfloor} (\delta \xi^{r-1})_{i_0, \ldots, i_r} = \int_\sigma \omega - \sum_{J: j_0 < \cdots < j_{r-1}} sign(J) \psi(J, \sigma, \omega).
$$

Therefore,

$$
\sum_\sigma \varepsilon_\sigma (\delta \xi^{r-1})_{i_0, \ldots, i_r} = (-1)^{\lfloor \frac{r}{2} \rfloor} \sum_\sigma \varepsilon_\sigma \left(\int_\sigma \omega - \sum_{J: j_0 < \cdots < j_{r-1}} sign(J) \psi(J, \sigma, \omega) \right)
$$

Since a is a cycle and ω is an exact form it follows that $\sum_\sigma \varepsilon_\sigma \int_\sigma \omega = \int_a \omega = 0$. By Proposition 3.18 we have

$$
(3.17) \qquad 0 = \partial a = \sum_\sigma \varepsilon_\sigma \sum_{J: j_0 < \cdots < j_{r-1}} sign(J)(i_{j_0}(\sigma), \ldots, i_{j_{r-1}}(\sigma))
$$

and thus,

$$(3.18) \qquad \sum_\sigma \varepsilon_\sigma \sum_{J:j_0<\cdots<j_{r-1}} sign(J)\psi(J,\sigma,\omega) = 0.$$

Formula (3.18) can be explained as follows. We can define an $(r-1)$ co-chain g_ψ as follows. For any $(r-1)$ simplex $\tau = (i_0,\ldots,i_{r-1})$ define

$$g_\psi(\tau) := \psi((0,\ldots,r),\sigma,\omega)$$

where $\sigma = (i_0,\ldots,i_{r-1},i_r)$ is any r simplex containing τ in its boundary. Co-chain g_ψ is well defined since ψ depends only only on $(i_0(\sigma),\ldots,i_{r-1}(\sigma)) = \tau$. Any pair (σ, J) where σ is an r-simplex and J is a permutation defines an $(r-1)$-simplex $\tau(\sigma, J) := (i_{j_0}(\sigma),\ldots,i_{j_{r-1}}(\sigma))$. Note that $g_\psi(\tau(\sigma, J)) = \psi(J,\sigma,\omega)$. Now, from (3.17) it follows that

$$
\begin{aligned}
(3.19) \qquad 0 &= g_\psi(\partial a) \\
&= g_\psi\left(\sum_\sigma \varepsilon_\sigma \sum_{J:j_0<\cdots<j_{r-1}} sign(J)(i_{j_0}(\sigma),\ldots,i_{j_{r-1}}(\sigma))\right) \\
&= \sum_\sigma \varepsilon_\sigma \sum_{J:j_0<\cdots<j_{r-1}} sign(J)g_\psi(\tau(\sigma, J)) \\
&= \sum_\sigma \varepsilon_\sigma \sum_{J:j_0<\cdots<j_{r-1}} sign(J)\psi(J,\sigma,\omega).
\end{aligned}
$$

$\square$

We turn now to the proof of Lemma 3.17 which is a consequence of the next formula.

Lemma 3.20. *In the above notation we have the following formula*

$$(3.20) \qquad \sum_{J:j_0<\cdots<j_t} sign(J)\int_{\partial Sd_{t+1}(J)} \xi^t_{i_{j_0},\ldots,i_{j_t}} =$$

$$(-1)^{t+1}\sum_{J:j_0<\cdots<j_{t+1}} sign(J)\int_{\partial Sd_{t+2}(J)} \xi^{t+1}_{i_{j_0},\ldots,i_{j_{t+1}}} +$$

$$(-1)^{r-t}\sum_{J:j_0<\cdots<j_{r-1}} sign(J)\sum_{K:k_0<\cdots<k_t} sign(K)\int_{Sd_{t+1}(J'_K)} \xi^t_{i_{j_{k_0}},\ldots,i_{j_{k_t}}}.$$

Proof. Let $J = (j_0,\ldots,j_r)$ be fixed with $j_0 < \cdots < j_t$. Set

$$\alpha(J,t,s) := \left((i_{j_0},\ldots,i_{j_t})^b,\ldots,(i_{j_0},\widehat{\ldots,i_{j_s}})^b,\ldots(i_{j_0},\ldots,i_{j_r})^b\right),$$

where $t \le s \le r$. We may represent

$$\partial Sd_{t+1}(J) = \sum_{s=t}^{r}(-1)^{s-t}\alpha(J,t,s)$$

and thus we may write the left hand side of (3.20) as

$$(3.21) \qquad \sum_{J:j_0<\cdots<j_t} sign(J) \int_{\partial Sd_{t+1}(J)} \xi^t_{i_{j_0},\ldots,i_{j_t}} \quad =$$

$$\sum_{J:j_0<\cdots<j_t} sign(J) \sum_{s=t}^{r}(-1)^{s-t} \int_{\alpha(J,t,s)} \xi^t_{i_{j_0},\ldots,i_{j_t}}.$$

Note that when $t < s < r$ there exist exactly two different permutations $J^{(1)} = J = (j_0,\ldots,j_r)$ and $J^{(2)} = (j_0,\ldots,j_{s-1},j_{s+1},j_s,j_{s+2},\ldots,j_r)$ such that $\alpha(J^{(1)},t,s) = \alpha(J^{(2)},t,s)$ also observe that $sign(J^{(1)}) = -sign(J^{(2)})$. It follows that

$$\sum_{J:j_0<\cdots<j_t} sign(J) \sum_{s=t}^{r}(-1)^{s} \int_{\alpha(J,t,s)} \xi^t_{i_{j_0},\ldots,i_{j_t}} \quad =$$

$$\sum_{J:j_0<\cdots<j_t} sign(J) \left(\int_{\alpha(J,t,t)} \xi^t_{i_{j_0},\ldots,i_{j_t}} + (-1)^{r-t} \int_{\alpha(J,t,r)} \xi^t_{i_{j_0},\ldots,i_{j_t}} \right) \quad =$$

$$\sum_{J:j_0<\cdots<j_t} sign(J) \left(\int_{Sd_{t+2}(J)} \xi^t_{i_{j_0},\ldots,i_{j_t}} + (-1)^{r-t} \int_{Sd_{t+1}(J')} \xi^t_{i_{j_0},\ldots,i_{j_t}} \right).$$

First we show that

$$(3.22) \qquad \sum_{J:j_0<\cdots<j_t} sign(J) \int_{Sd_{t+2}(J)} \xi^t_{i_{j_0},\ldots,i_{j_t}} \quad =$$

$$(-1)^{t+1} \sum_{J:j_0<\cdots<j_{t+1}} sign(J) \int_{\partial Sd_{t+2}(J)} \xi^{t+1}_{i_{j_0},\ldots,i_{j_{t+1}}}.$$

If t is fixed and $J = (j_0,\ldots,j_r)$ is a permutation denote by (J_s) the permutation $((J_s)_0,\ldots,(J_s)_r)$ where

$$(J_s)_l = \begin{cases} j_l & \text{if } 0 \leq l \leq s-1 \\ j_{l+1} & \text{if } s \leq l \leq t \\ j_s & \text{if } l = t+1 \\ j_l & \text{if } t+2 \leq l \leq r \end{cases},$$

where $0 \leq s \leq t+1$. Permutation (J_s) is obtained from J by removing j_s and replacing it between j_{t+1} and j_{t+2}. Note that if $s = t+1$ then $(J_s) = J$. Observe that

$$(3.23) \qquad Sd_{t+2}(J) = Sd_{t+2}((J_s))$$

since $\{(J_s)_0,\ldots,(J_s)_{t+1}\} = \{j_0,\ldots,j_{t+1}\}$. There exists a 1:1 and onto correspondence

$$f : \{(J,s) : s = 0,\ldots,t+1, \quad j_0 < \cdots < j_{t+1}\} \to \{J : j_0 < \cdots < j_t\},$$

given by $(J,s) \mapsto (J_s)$.

First we show that f is $1:1$. Suppose that (J^1,s^1) and (J^2,s^2) are two different elements that are mapped to $\tilde{J} = (\tilde{j}_0,\ldots,\tilde{j}_r)$. So, $\tilde{j}_{t+1} = j^1_{s^1} = j^2_{s^2}$. But

since $\tilde{j}_0 < \cdots < \tilde{j}_r$ there exists unique l, $0 \leq l \leq t$, such that $\tilde{j}_l < \tilde{j}_{t+1} < \tilde{j}_{l+1}$ if $l < t$ and $\tilde{j}_t < \tilde{j}_{t+1}$ if $l = t$. That is, there exists a unique place between the elements of the sequence $\tilde{j}_0, \ldots, \tilde{j}_t$ where $\tilde{j}_{t+1}$ can be squeezed to form an ordered sequence. It follows that $(J^1, s^1) = (J^2, s^2)$.

The correspondence f is onto. Let $J = (j_0, \ldots, j_r)$, $j_0, \ldots, j_t$. If $j_{t+1} > j_t$ then set $s := t + 1$. Otherwise, there exists a unique s such that $j_{s-1} < j_{t+1} < j_s$. Therefore, $J = f(\tilde{J}, s)$ where

$$\tilde{J}_l = \begin{cases} j_l & \text{if } 0 \leq l \leq s - 1 \\ j_{t+1} & \text{if } l = s \\ j_{l-1} & \text{if } s + 1 \leq l \leq t + 1 \\ j_l & \text{if } t + 2 \leq l \leq r \end{cases}.$$

Permutation $\tilde{J}$ is obtained from J by swapping j_s with j_{t+1}. Next we use correspondence f in the left hand side of formula (3.22)

$$\sum_{J: j_0 < \cdots < j_t} sign(J) \int_{Sd_{t+2}(J)} \xi^t_{i_{j_0}, \ldots, i_{j_t}} =$$

$$\sum_{J: j_0 < \cdots < j_{t+1}} \sum_{s=0}^{t+1} sign((J_s)) \int_{Sd_{t+2}((J_s))} \xi^t_{i_{j_0}, \ldots \hat{i}_{j_s} \ldots, i_{j_{t+1}}} =$$

$$\sum_{J: j_0 < \cdots < j_{t+1}} \sum_{s=0}^{t+1} sign(J)(-1)^{t-s+1} \int_{Sd_{t+2}(J)} \xi^t_{i_{j_0}, \ldots \hat{i}_{j_s} \ldots, i_{j_{t+1}}} =$$

$$(-1)^{t+1} \sum_{J: j_0 < \cdots < j_{t+1}} sign(J) \int_{Sd_{t+2}(J)} (\delta \xi^t)_{i_{j_0}, \ldots, i_{j_{t+1}}} =$$

$$(-1)^{t+1} \sum_{J: j_0 < \cdots < j_{t+1}} sign(J) \int_{\partial Sd_{t+2}(J)} \xi^{t+1}_{i_{j_0}, \ldots, i_{j_{t+1}}}.$$

where in the second equality we used equation (3.23) and $sign((j_s)) = (-1)^{t-s+1} sign(J)$ since it takes $t - s + 1$ transpositions to transform J into (J_s). In the third equality we used the definition of $\delta \xi^t$ and in the last equality we used Stokes formula after using a relation defining ξ^{t+1} (see Definition 3.8).

Next we prove

(3.24)
$$\sum_{J: j_0 < \cdots < j_t} sign(J) \int_{Sd_{t+1}(J')} \xi^t_{i_{j_0}, \ldots, i_{j_t}} =$$

$$\sum_{J: j_0 < \cdots < j_{r-1}} sign(J) \sum_{K: k_0 < \cdots < k_t} sign(K) \int_{Sd_{t+1}(J'_K)} \xi^t_{i_{j_{k_0}}, \ldots, i_{j_{k_t}}}.$$

There exists a 1:1 and onto correspondence

$$h : \{(J, K) : j_0 < \cdots < j_{r-1}, \quad k_0 < \cdots < k_t\} \to \{J : j_0 < \cdots < j_t\},$$

defined by $(J, K) \mapsto (j_{k_0}, \ldots, j_{k_{r-1}}, j_r) = (J'_K, j_r)$.

h is 1:1. Suppose that (J^1, K^1) and (J^2, K^2) are mapped to $(j_0, \ldots, j_r)$. Then

we must have $j_r = j_r^1 = j_r^2$. It follows that $\{j_0^1, \ldots, j_{r-1}^1\} = \{j_0^2, \ldots, j_{r-1}^2\}$. But since $j_0^l < \cdots < j_{r-1}^l$ for $l = 1, 2$ we must have $J^1 = J^2$. Since K^1 and K^2 are permutations, $(J^1)'_{K^1} = (J^2)'_{K^2}$ and $(J^l)'$, $l = 1, 2$ have distinct elements it follows that $K^1 = K^2$.

h *is onto.* Let J, $j_0 < \cdots < j_t$ be a permutation. Set J^1 to be a permutation with $j_0^1 < \cdots < j_{r-1}^1$, $j_r^1 := j_r$ and $\{j_0, \ldots, j_{r-1}\} = \{j_0^1, \ldots, j_{r-1}^1\}$. There exists a permutation K of $\{0, \ldots, r-1\}$ such that $j_i = j_{k_i}^1$. Now, $(J^1, K) \mapsto J$.

We apply correspondence h to left hand side of (3.24) obtaining

$$\sum_{J: j_0 < \cdots < j_t} sign(J) \int_{Sd_{t+1}(J')} \xi^t_{i_{j_0}, \ldots, i_{j_t}} \quad =$$

$$\sum_{J: j_0 < \cdots < j_{r-1}} \sum_{K: k_0 < \cdots < k_t} sign(J'_K, j_r) \int_{Sd_{t+1}(J'_K)} \xi^t_{i_{j_{k_0}}, \ldots, i_{j_{k_t}}}.$$

Finally, note that $sign(K)sign(J) = sign(J'_K, j_r)$ and the lemma is proven.   □

We are ready now to prove Lemma 3.17.

Proof of Lemma 3.17. The proof is by induction on m. For $m = 0$ we have

$$\int_\sigma \omega = \sum_J sign(J) \int_{Sd_1(J)} d\xi^0_{i_{j_0}} = \sum_J sign(J) \int_{\partial Sd_1(J)} \xi^0_{i_{j_0}}.$$

Suppose that $m > 0$. Induction hypothesis for $m - 1$ followed by application of Lemma 3.20 yields

$$\int_\sigma \omega \;=\; (-1)^{\lfloor \frac{m}{2} \rfloor} \sum_{J: j_0 < \cdots < j_{m-1}} sign(J) \int_{\partial Sd_m(J)} \xi^{m-1}_{i_{j_0}, \ldots, i_{j_{m-1}}}$$

$$+ \quad (-1)^r \sum_{J: j_0 < \cdots < j_{r-1}} sign(J) \sum_{t=0}^{m-2} (-1)^{\lfloor \frac{t}{2} \rfloor} \sum_{K: k_0 < \cdots < k_t} sign(k) \int_{Sd_{t+1}(J'_K)} \xi^t_{i_{j_{k_0}}, \ldots, i_{j_{k_t}}}$$

$$= \quad (-1)^{\lfloor \frac{m}{2} \rfloor} \left[(-1)^m \sum_{J: j_0 < \cdots < j_m} sign(J) \int_{\partial Sd_{m+1}(J)} \xi^m_{i_{j_0}, \ldots, i_{j_m}} \right.$$

$$+ \quad (-1)^{r-(m-1)} \sum_{J: j_0 < \cdots < j_{r-1}} sign(J) \sum_{K: k_0 < \cdots < k_{m-1}} sign(K) \int_{Sd_m(J'_K)} \xi^t_{i_{j_{k_0}}, \ldots, i_{j_{k_{m-1}}}} \left. \right]$$

$$+ \quad \sum_{J: j_0 < \cdots < j_{r-1}} sign(J) \sum_{t=0}^{m-2} (-1)^{\lfloor \frac{t}{2} \rfloor} \sum_{K: k_0 < \cdots < k_t} sign(k) \int_{Sd_{t+1}(J'_K)} \xi^t_{i_{j_{k_0}}, \ldots, i_{j_{k_t}}}$$

$$= \quad (-1)^{\lfloor \frac{m+1}{2} \rfloor} \sum_{J: j_0 < \cdots < j_m} sign(J) \int_{\partial Sd_{m+1}(J)} \xi^m_{i_{j_0}, \ldots, i_{j_m}}$$

$$+ \quad (-1)^r \sum_{J: j_0 < \cdots < j_{r-1}} sign(J) \sum_{t=0}^{m-1} (-1)^{\lfloor \frac{t}{2} \rfloor} \sum_{K: k_0 < \cdots < k_t} sign(k) \int_{Sd_{t+1}(J'_K)} \xi^t_{i_{j_{k_0}}, \ldots, i_{j_{k_t}}},$$

where the last equality is since the parity of $\lfloor \frac{m}{2} \rfloor + m =$ parity of $\lfloor \frac{m+1}{2} \rfloor$ and the parity of $(m-1) + \lfloor \frac{m}{2} \rfloor =$ parity of $\lfloor \frac{m-1}{2} \rfloor$ and the Lemma is proven. $\square$

4. $L^{\overline{p}}$-COHOMOLOGY

In this section we study an $L^{\overline{p}}$ complex, introduced in [GoTr]. We show that that Poincaré inequality for differential forms on a manifold X, implies an embedding of $L^{\overline{p}}$-cohomology of X into the standard De Rham cohomology. The main tool we use here is a method developed by B. Yousin [Y] to regularize the forms.

Definition 4.1. Let $\overline{p} = (p_0, \ldots, p_n)$, $1 \le p_i \le \infty$, be an n-tuple of numbers. Set

$$\Omega_{\overline{p}}^k(X) := \{\omega \in L^{p_k}(\Omega^k) : d\omega \in L^{p_{k+1}}(\Omega^{k+1})\}.$$

The complex $(\Omega_{\overline{p}}^\bullet(X), \overline{d})$ where $\overline{d}$ is the weak exterior derivative, is called $L^{\overline{p}}$ **De Rham complex**.

The cohomology ring of $(\Omega_{\overline{p}}^\bullet(X), \overline{d})$ is denoted by $H_{\overline{p}}^\bullet(X)$ and called $L^{\overline{p}}$ **cohomology** of X.

We say that the complex $\Omega_{\overline{p}}^\bullet(X)$ satisfies generalized Poincaré inequality if there exist positive constants C_k such that for each k-form ω such that $d\omega \in L^{p_{k+1}}$ there exists a closed k-form ω_0 such that

$$\|\omega - \omega_0\|_{p_k} \le C_k \|d\omega\|_{p_{k+1}}.$$

Theorem 4.2. *([Y] Theorem 2.7.1) Let X be a smooth Riemaniann manifold and $\omega \in L^p(X)$ a $k+1$-form with $\overline{d}\omega \in \mathcal{C}^\infty(X)$. Then for any $\varepsilon > 0$ there exists a k-form ψ_ε such that*

 (1) $\|\psi_\varepsilon\|_p < \varepsilon$,
 (2) $\|\overline{d}\psi_\varepsilon\|_p < \varepsilon$,
 (3) $\omega + \overline{d}\psi_\varepsilon$ is smooth.

Corollary 4.3. *Let $\Omega_{\overline{p}}^\bullet(X)$ be an $L^{\overline{p}}$ complex that satisfies Poincaré inequality. Suppose that $\omega \in \Omega_{\overline{p}}^{k+1}$ and $d\omega$ are smooth on X. Then, there exists a k-form ψ such that*

 (1) $\|\psi\|_{p_k} < \infty$,
 (2) $\|\overline{d}\psi\|_{p_{k+1}} < \infty$,
 (3) $\omega + \overline{d}\psi$ is smooth.

Proof. By Theorem 4.2 there exists ψ_0 such that $\|\psi_0\|_{p_{k+1}} < \infty$, $\|\overline{d}\psi_0\|_{p_{k+1}} < \infty$ and $\omega + \overline{d}\psi_0$ is smooth. By generalized Poincaré inequality there exists a closed k-form ψ_1 such that

$$\|\psi_0 - \psi_1\|_{p_k} \lesssim \|\overline{d}\psi_0\|_{p_{k+1}}.$$

Set

$$\psi := \psi_0 - \psi_1.$$

Now, $\overline{d}\psi = \overline{d}\psi_0$ so $\omega + \overline{d}\psi$ is smooth. We also have

$$\|\psi\|_{p_k} = \|\psi_0 - \psi_1\|_{p_k} \lesssim \|\overline{d}\psi_0\|_{p_{k+1}} < \infty.$$

$\square$

Theorem 4.4. *There is a natural inclusion $i : H_{\overline{p}}^{\bullet}(X) \to H_{DR}^{\bullet}(X)$ where $H_{DR}^{\bullet}(X)$ is the De Rham cohomology group.*

Proof. First we show existence of a map $i : H_{\overline{p}}^{\bullet}(X) \to H^{\bullet}(X)$. Let ω be a closed form in $\Omega_{\overline{p}}^{k+1}(X)$ then by Corollary 4.3 there exits ψ such that $\alpha := \omega + \overline{d}\psi$ is smooth, $\|\psi\|_{p_k} < \infty$ and $\|d\psi\|_{p_{k+1}} < \infty$. So α is smooth and defines the same cohomology class as ω. Set $i[\omega]$ to be the cohomology class of α in $H^{k+1}(X)$. We claim that the map i is well defined. First we show that i is independent of choice of the form ψ. Suppose that ψ' is another form such that $\alpha' := \omega + \overline{d}\psi'$ is smooth, $\|\psi'\|_{p_k} < \infty$ and $\|d\psi'\|_{p_{k+1}} < \infty$. We have

$$\alpha - \alpha' = \overline{d}(\psi - \psi')$$

Since $\overline{d}(\psi - \psi')$ is smooth, there exists a form $\xi \in \Omega_{\overline{p}}^{k-1}$ such that $\psi - \psi' + d\xi$ is smooth and therefore, $\alpha - \alpha' = d(\psi - \psi' + d\xi)$.

Next, we show that i is independent of the representative ω. Suppose that ω'' is another form from the cohomology class of ω in $H_{\overline{p}}^{k+1}(X)$. Since we proved independence from ψ, we may assume that ω and ω'' are smooth. Then there exists a form $\gamma \in \Omega_{\overline{p}}^{k}$ such that $\omega - \omega'' = d\gamma$. Since $d\gamma$ is smooth we may find a form β such that $\gamma + \overline{d}\beta$ is smooth. The claim follows.

Finally we show injectivity of i. Suppose that $\omega \in \Omega_{\overline{p}}^{k+1}$ is a smooth closed form such that $\omega = d\gamma$ where γ is smooth. By generalized Poincaré inequality there exists a closed form γ_0 such that

$$\|\gamma - \gamma_0\|_{p_k} \lesssim \|\omega\|_{p_{k+1}}.$$

It follows that $\gamma - \gamma_0 \in \Omega_{\overline{p}}^{k}$ and $d(\gamma - \gamma_0) = \omega$.

$\square$

Note that the inclusion in Theorem 4.4 is not surjective in general.

Example 4.5. Let X be an open punctured disk in $\mathbb{R}^2$ and $\overline{p} := (p, p, p)$. It is well known that $H_{DR}^1(X)$ is one dimensional and spanned by the angle form $\omega = \frac{xdy - ydx}{x^2 + y^2}$ where x, y are the standard coordinates in $\mathbb{R}^2$. A straight forward computation shows that for $p \geq 2$ the form ω is not L^p bounded and therefore $H_{\overline{p}}^1(X) = 0$. Indeed, suppose that α is a smooth 1-form representing a non trivial element of $H_{\overline{p}}^1(X)$. Since the disk X can be covered by three sectors each of which convex, it follows that X satisfies Poincaré inequality. Therefore, there exists a non zero number a such that $\int_{c_r} \alpha = a$ for any circle c_r of radius r around the origin. It follows that

$$|a| = |\int_{c_r} \alpha| \leq \int_{c_r} |\alpha| \leq (\int_{c_r} |\alpha|^p)^{1/p}(2\pi r)^{1/p'}$$

and hence

$$\int_{c_r} |\alpha|^p \geq \frac{|a|^p}{(2\pi r)^{p-1}}.$$

Therefore, writing the norm of α in polar coordinate (r, θ) we obtain

$$\|\alpha\|_{L^p}^p = \int_0^1 dr \int_{c_r} |\alpha|^p d\theta \geq \frac{|a|^p}{(2\pi)^{p-1}} \int_0^1 \frac{dr}{r^{p-1}} = \infty,$$

in contradiction with the fact that $\|\alpha\|_{L^p} < \infty$.

As a consequence of Theorem 4.4 we obtain the following

Corollary 4.6. *If X is a compact manifold then $i : H_{\overline{p}}^\bullet(X) \to H_{DR}^\bullet(X)$ is an isomorphism*

Proof. By Theorem 4.4 the map i is injective so we only need to show that it is surjective. But a smooth form on compact manifold is bounded and therefore L^p bounded. $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, 40 ST. GEORGE ST, TORONTO, ON, CANADA M5S 2E4

E-mail address: shartl@math.toronto.edu